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Teaching Prospective Teachers about Mathematical Reasoning: An Example from Practice

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Abstract

Mathematical reasoning in the K-12 classroom includes the teacher helping students share, explain, test, and revise mathematical ideas. While mathematical reasoning has long been established as a key component for effectively teaching and learning mathematics, understanding what teachers can do and what they should know to promote their students' mathematical reasoning remains underdeveloped. This article unpacks an example of engaging prospective K-8 teachers with mathematical reasoning in an undergraduate mathematics course to help them learn about mathematical reasoning and to illustrate how they might teach it to K-8 students. How the prospective teachers responded to the experience is discussed, as well as their perceptions of K-8 students' mathematical reasoning capabilities. Implications for teaching about mathematical reasoning as part of teacher development are also proposed.

> Mathematical reasoning develops in classrooms where students are encouraged to put forth their own ideas for examination. Teachers and students should be open to questions, reactions, and elaborations from others in the classroom. Students need to explain and justify their thinking and learn how to detect fallacies and critique others' thinking. (National Council of Teachers of Mathematics [NCTM], 2000, p. 188)

> Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science (Polya, 1957, p. vii).

Sharing, explaining, testing, and revising mathematical ideas is fundamental to doing and creating mathematics (Lakatos, 1989). Before a mathematical formula, algorithm, or theorem and proof appears in a mathematics textbook, research journal, or conference presentation, it has been fiddled with, tweaked, illustrated with (counter) examples, perhaps set aside and then revisited, and shared with others in the discipline (Polya, 1957). This process of doing mathematics is consistent with findings from educational psychology that learning occurs within a context of social interaction and communication and shapes what we know (Vygotsky, 1978). In other words, the mathematics that we learn is intertwined with how we learn it; memorizing definitions, rules, and procedures results in learning thin mathematics that is different than deeper mathematics learned through problem solving and mathematical reasoning within a community of learners (Schoenfeld, 1985).

Mathematical reasoning, a process where mathematical ideas are shared, explained, tested, and revised in a disciplinary and/or learning community, has been recognized as a key component of effective K-12 mathematics teaching and learning for over fifty years (cf. Conklin, Grant, Ludema, Rickard, & Rivette, 2006; Fitzgerald, Winter, Lappan, & Phillips, 1986; Lamberg, 2013; NCTM, 2000; Polya, 1957). In identifying mathematical reasoning as central to teaching and learning mathematics, reforms reflect what Bruner (1960) described as intellectual honesty – students should have opportunities to learn about the discipline as it is actually done. For example, NCTM (1991) stresses the need for both the teacher and students to play active roles in learning mathematics, and that learning how to reason with mathematics is critical. While recent research has noted various interpretations of mathematical discourse and reasoning in the classroom (e.g., Herbal-Eisenmann & Otten, 2011; Ryve, 2011; Truxaw & DeFranco, 2008), the predominant picture that emerges is that mathematical reasoning should be integral to the mathematics classroom and necessarily includes students talking about mathematics with the informed support of their teacher (Lamberg, 2013; Rickard, 1995, 1996, 1998).

But while K-12 students developing skills in mathematical reasoning is clearly important, how teachers contribute to making this happen is less clear. For example, teachers need strong subject matter knowledge to help their students unpack a mathematical idea in multiple ways (e.g., Wilson, Shulman, & Richert, 1987),[1] but the unpredictability of students' ideas also requires teachers to be flexible to help students revise flawed mathematics (e.g., Rickard, 2014). The uneven terrain of students' mathematical reasoning requires teachers to navigate with students to test and refine their ideas and get back on track, and how teachers accomplish this with their students is varied and uncertain (e.g., Rickard, 2005a, 2005b). Research going back over twenty years has examined how teachers facilitate mathematical reasoning with their students and dilemmas or challenges they may encounter (cf. Ball, 1990; Conklin et al., 2006; Rickard, 2014). However, what teachers need to know and be able to do in their classrooms with their students to provide effective learning experiences in mathematical reasoning is less developed (Lamberg, 2013).

Mathematical Reasoning in Mathematics and in Teaching

Proofs and Refutations

Lakatos (1989) describes mathematical reasoning, as it occurs in the discipline of mathematics, as "proofs and refutations." In this process, a new idea is proposed and shared within the mathematics community, then explained more carefully through an informal argument or proof, then tested by the community using examples and/or counter examples, and finally revised or formally proved (Lakatos, 1989). Revisions may consist of modifying the original idea based on how it withstands the examples and/or counter examples. Whether the venue is a graduate mathematics seminar, a presentation at a research conference, or an undergraduate or K-12 classroom, Lakatos argues that the proofs and refutations process is a reasonable framework for how mathematical reasoning progresses to understand and create mathematics.

[1] Wilson et al. (1987) called this *pedagogical content knowledge*, which is a kind of subject matter knowledge specific to teaching, and includes knowledge of representations, examples, and explanations that are tailored to helping students learn particular concepts and/or procedures.

Interestingly, Lakatos's proofs and refutations framework is similar to the heuristic Polya (1957) developed for problem solving: understand the problem, devise a plan, carry out the plan, look back/revise. A key aspect, in which the frameworks differ, however, is that Polya's heuristic is about finding a solution to a stated problem, whereas Lakatos's proofs and refutations is a framework for creating and proving mathematical ideas (cf. Lakatos, 1989; Polya, 1957).

An Example of Mathematical Reasoning with Proofs and Refutations

As a university faculty member and mathematics educator, I use the proofs and refutations framework to structure experiences for my undergraduate students that will engage them with and help them learn about mathematical reasoning.[2] My further intent is that these experiences will inform their future teaching with their own K-8 students. For one such example of learning about mathematical reasoning, I proposed the following idea to my class, which is modified from an item that Ball (1988) used to explore the mathematics subject matter knowledge that prospective teachers bring to teaching:

> *One of your students is very excited and says that she has come up with a new mathematical theory. She says that as the perimeter of a figure increases, then its area also increases. She shows you these diagrams that she says prove her theory and she wants to share it with the class:*

W = 2 L = 4

P = 12; A = 8

W = 4 L = 8

P = 24; A = 32

> *"Look," she says, "at how the area increases when the perimeter gets bigger!"*
> *What do you think of the student's theory and how would you respond to her?*

After presenting this item, I asked my students to think about the mathematics of the student's theory and their response to her, and then we shared our thinking.[3]

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[2] I teach a two-semester course sequence in mathematics (Mathematics for Elementary Teachers I and II) that is required for undergraduate students in the Bachelor of Arts in Education major, so all students in my mathematics courses are prospective teachers.

[3] Using the item, Ball (1988) found that prospective teachers' subject matter knowledge was thin and that many agreed with the student's theory. In my undergraduate course, I focus on using the item to unpack mathematical reasoning as suggested by the Lakatos (1989) proofs and refutations framework and getting my students to engage with mathematical reasoning and think about how they might teach it.

Presentation of the above item is the first stage of the proofs and refutations process.[4] After allowing my students a few minutes to think about the student's proposed theory, we moved on to a careful explanation, illustrated by additional examples. First, I asked for a volunteer to explain how the example shown in the item demonstrated the student's theory. After raising her hand, one of my students explained that, "Her example shows that both the perimeter and area got bigger, so as the perimeter increased, so did the area." Seeing nods of agreement from the class, I next asked, "OK -- how many people agree with the student's theory?" About one-third of my class of 27 raised their hands and, after two more follow-up questions regarding who disagreed and who wasn't sure, about half the class disagreed with the student's theory and the remainder indicated that they were not sure.

Moving on to the next step of the proofs and refutations framework, I asked those who agreed with the student's conjecture to illustrate the theory with some additional explanation as to why they believed it is correct or even describe a proof. One supporter of the theory shared that, "It makes sense because for the perimeter of the rectangle to get bigger, the length and width have to increase, and since the area is the multiplied length and width it has to get larger too." Other supporters of the student's theory nodded in agreement, one adding that the theory works for other figures too, such as circles. He noted that, as with rectangles, "If the radius or diameter of a circle gets bigger then the circumference – or perimeter – of the circle gets bigger and so does the area." At this point, I asked the class, whether they individually agree with the student's theory or not, or weren't sure, if they understood what it was saying. Everyone in the class nodded and I emphasized that when working with K-8 students in their own classrooms, it's important to make sure that the mathematical idea be understood so that it can be explored and tested further.

At the third stage of the proofs and refutations process, members of the mathematics community unpack a mathematical idea further by testing it with additional examples or asking further questions. So now I noted to the class that, so far, we've had some examples that illustrate the theory and seem to support it. What about examples from those who don't think the student's theory is valid – i.e., counter examples? One of the students, who had raised her hand indicating that she did not think the student's theory was correct, raised her hand again and said, "I have a counter example" and I asked her to share it on the whiteboard. She shared the following:

W = 2 L = 4 — *W = 1 L = 7*

P = 12; A = 8 — *P = 16; A = 7*

After sketching the above rectangles, I asked if everyone agreed with the calculations for the perimeter and area of each rectangle and all nodded. I then asked the student why she had called this a counter example. She explained, "Well, the perimeter of the rectangles increased – from 12 to 16 – but the area didn't increase, it went down – from 8 down to 7 square units." She added

[4] I did not discuss the Lakatos (1989) proofs and refutations framework explicitly with my class, but instead use it as a framework for my own planning and structuring the class discussion.

further that, "So this doesn't follow the theory because the student said in her theory that as the perimeter increases the area increases, but here the area didn't increase it went down." Looking around the room, some students were nodding in agreement while others appeared puzzled.

Thanking the student for sharing her counter example, I asked if there were any questions about why the counter example did not conform to the theory. Seeing no indication of questions, I then asked for any comments from students who had initially felt that the theory was valid. The same student who had supported the theory and proposed the example of circles having increased area when the perimeter (or circumference) increased, said that, "I see it doesn't work all the time for rectangles, but I still think that it has to work for circles." Noting that several students where nodding in agreement, I asked the class how many thought that the theory worked for circles even if it didn't work for rectangles and almost everyone raised their hands. At this point in the lesson, I felt that we were ready to move into the fourth stage of the proofs and refutations framework, which is to revise the theory in the mathematics community. I said to the class,

> OK. So we started with a mathematical idea that many of us thought was valid, but working together we see that it doesn't always work for some shapes – like rectangles – but it may work for others, like circles. What mathematicians often do in a situation like this is try to revise the theory to try to come up with something that is true.[5]

I then asked everyone to take a few minutes and think about how the theory could be revised so that it was true. I added that, "One way to help us to come up with a revised theory is to find more examples where it works and doesn't work and then try to compare them and maybe find some patterns."

After a few minutes, the student who had earlier argued that the original theory worked for circles raised his hand again and said, "I still think it works for circles, so can we just say, 'as the perimeter of a circle increases, its area will also increase'?" I paused to write the circle conjecture on the whiteboard and then asked the class, "What do we think – does this make sense?" Another student, who had not spoken to the class before, raised her hand and said, "I know that it didn't work for rectangles, but I think it does work for squares, even though they're rectangles." As I wrote "squares" on the whiteboard, another student raised his hand and contributed that the theory also worked for equilateral triangles, "Which are to triangles like squares are to rectangles," he said. When I asked him to explain this idea further, he said that the theory "wouldn't work for any triangle, just like it didn't work for any rectangle," but would work for equilateral triangles because, like the square, "all the sides are equal." As I wrote "equilateral triangles" on the whiteboard, I said to the class, "OK – so far we seem to think that the theory works for circles, squares, and equilateral triangles, does anyone disagree with that?" No hands were raised, so we had class consensus that the while the theory was not valid for all figures as originally proposed, it is valid for circles, squares, and equilateral triangles.

Continuing with trying to revise the theory as a mathematical community, I stated to the class that we should probably think about two things now: Are there other figures for which the theory is valid besides circles, squares, and equilateral triangles and, if so, do that have something in common; and, secondly, how might we revise the original theory to account for the

[5] Lakatos (1989) referred to revising a conjecture or theory to accommodate counter examples as "monster barring" and noted its wide use in mathematics.

figures and/or their commonalities? One student, making her first contribution to the class, said that she felt that, "There are lots of figures that the theory will work, they just have to have all sides equal – like a pentagon or hexagon, or whatever, will work so long as all the sides are equal." In rapid succession, another student commented that the theory works for "regular polygons, which is what the squares and equilateral triangles are" but then another student said, "But circles aren't polygons and the theory works for them too," followed by a third student who commented, "It's proportional – as long as the figures are proportional the theory will work." Interjecting myself into the discussion, I said, "Let's try to tie some of these things together – the class has identified other shapes, and maybe commonalities, for which the theory might be valid, so now we have to try to use this information to revise the theory." I then recorded "all sides equal," "regular polygons," and "proportionality" on the whiteboard with the other terms. I then asked the group to spend a few minutes thinking about how these findings might be used to revise the student's original theory into something we felt would be mathematically valid.

As I circulated around the room, I noticed that most students had sketches of various figures in their notes. All were engaged in writing more notes, speaking quietly with a peer, or both. As they worked, I commented to the group that, "Remember, what you're doing now, this is what we want our K-8 students to do also – talk about mathematics, compare ideas – 'no one of us is smarter than all of us'."[6] After another few minutes of work time, I asked the class for their attention and to please share any ideas that they had. The first student to raise his hand suggested, "Let's revise the theory so that it says, 'If the perimeter of a figure increases proportionally, then the area also increases'." I asked him to explain his reasoning further. He described how "proportionality is important because it means you can't increase one dimension and shrink another like in the counter example – you need to increase all the dimensions." Another student, making her first contribution to the discussion, noted that proportionality didn't need to be included in the revised theory – "as long as all the dimensions of a figure increase, the perimeter and area will both have to increase too," she said. Finally, another student noted that this idea explains why the theory works for squares and equilateral triangles, "Because for those all the dimensions have to increase for all the sides to stay the same." With nods of agreement all around the classroom, I asked what our final conjecture should be and the class settled on, *If the perimeter of a figure increases by increasing all its dimensions, then the area of the figure increases*.[7]

Reflection and Discussion

With the revised conjecture agreed upon, I shared with the class that what we had done

[6] "No one of us is smarter than all of us" was shared with me by Annie Blue of Togiak, Alaska. Annie Blue is a Yup'ik elder widely recognized throughout Alaska as a master storyteller and expert on Alaska Native culture (Kisker, Lipka, Adams, Rickard, Andrew-Irke, Yanez, & Millard, 2012; Lipka, Jones, Gilsdorf, Remick, & Rickard, 2010).

[7] The conjecture could be refined further – e.g., the process of proofs and refutations could be repeated. However, with class time running short and my students satisfied with having revised and improved the conjecture, I decided to move on.

together in sharing, explaining, testing, and revising the original theory "parallels"[8] how mathematics is actually done. By engaging our own K-8 students in mathematical reasoning, as in this example, K-8 students can not only learn mathematics but learn about doing mathematics. Bolstering the importance of substantive mathematical discussions in the K-8 classroom, I reminded my students that mathematical reasoning and proof is one of the five NCTM process standards (see NCTM, 2000). With the roughly 15 minutes we had left in the class, I asked my students what their overall thoughts were about our mathematical discussion. In general, the class seemed to think that the discussion was worthwhile and had a better sense of what goes on in creating mathematics. One student reflected that during the prior semester we had studied the Babylonian and Mayan numeration systems, and that the experience of unpacking and revising the student's theory gave her a sharper sense of how things like numeration systems may have developed through discussion and debate among people. Several students commented that having students develop, share, and explain their own examples (or counter examples) was really important and good for K-8 students in developing deeper understanding of mathematics, with one observing that sharing ideas is also a vehicle for student assessment.

Another interesting factor emerged from the discussion. Several students articulated, and some of the rest of the class agreed, that K-8 students might not be capable of the kind of mathematical reasoning and discussion that we had just had. As one skeptical student queried, "Can fourth or fifth graders really talk about stuff like this and come up with ideas like that?" The few moments we had remaining in the class were not sufficient to answer this question completely, but I firmly assured the class that, "Yes, K-8 students can have discussions like we have just had *with your help*." I reminded my students about the centrality of the teacher's role in structuring the mathematical reasoning and guiding the classroom discussion. Importantly, the conjecture or example serving as the catalyst for the discussion should be carefully chosen. Sometimes, rich mathematical starting points can emerge unexpectedly and are not anticipated (e.g., Rickard, 2014), but are often intentionally crafted by the teacher (e.g., Schoenfeld, 1985). Moreover, whether planned or spontaneous, I promised my students that, working with their own K-8 students, they would be pressed in balancing dilemmas of time, what mathematics is most worth reasoning about, and respecting K-8 students as members of a mathematics community of learners (Ball, 1990). Wrapping up our class for the day, I reminded my students that we had engaged in mathematical reasoning many times since the beginning of the academic year and had had many discussions about doing mathematics. But these previous experiences had usually focused on a specific problem and the goal was developing a problem solving approach to finding a solution to the problem.[9] In the case of the student's theory about perimeter and area of figures, we used a conjecture as a starting point and focused on understanding, testing, and revising it into a new, better, and valid *conjecture*, not necessarily a *solution*.

[8] This pun was intended and, as hoped for, it evoked groans from some of my students.

[9] An example would be the "handshake problem," which we worked on during the first day of the two course sequence: "If everyone in the room shakes hands with everyone else, how many handshakes are there? How many handshakes are there if there are n people in the room?"

Conclusion: Limitations and Implications of Teaching Mathematical Reasoning

This example of mathematical reasoning that I have shared is my account of my own experience with my own students. While I believe I have provided an accurate summary of the events and comments from my undergraduate class, I make no claims of objectivity or that our experience can be generalized to other professors or prospective teachers. Like other researchers who have studied their own practices, my goal is to unpack and gain insight into issues of practice, not to develop findings that can be generalized to others (e.g., Ball, 1990). Moreover, a variety of case studies have advanced understanding of teachers' and students' roles and contributions to teaching and learning problem solving and mathematical reasoning (e.g., Conklin et al., 2006; Lamberg, 2013; NCTM, 2000; Rickard, 2005a, 2005b, 2014). Similarly, the example of teaching my own students about mathematical reasoning using Lakatos's (1989) proofs and refutations process has potential to shed light on teaching prospective teachers about mathematical reasoning. The value of the example is not in providing prescriptions for professors or prospective teachers, but instead a rough sketch or outline of what teaching and learning mathematical reasoning might look like – which must then be filled in with details specific to the needs and strengths of particular teacher and learners.

I believe the example from my class demonstrates that prospective K-8 teachers can readily engage in mathematical reasoning. Specifically, I was able to coach my class through the steps of Lakatos's (1989) proofs and refutations framework as they wrestled with a conjecture relating the perimeter and area of figures. The conjecture was selected because I believed it would be challenging to my students, yet also incorporates accessible mathematics that are topics in the K-8 mathematics curriculum (see Ball, 1988). Using Lakatos's (1989) process of proofs and refutations as a framework for our mathematical reasoning experience was useful as it has the potential to serve as a framework for my students in their own teaching. In our class reflection and discussion, we explicitly noted that we had gone through a process of sharing, explaining, testing, and revising a student's conjecture – this is shorthand for the proofs and refutations process and can be used as a heuristic by my K-8 students in their own planning and thinking about teaching mathematical reasoning.

While the example shows that our class experience with mathematical reasoning was beneficial, it is unlikely by itself to be sufficient in preparing prospective K-8 teachers to teach mathematical reasoning. In fact, like other aspects of teaching and learning mathematics, effectively teaching mathematical reasoning likely requires ongoing professional development at both preservice and inservice stages of teacher development (Lamberg, 2013; NCTM, 1991). Integrating multiple opportunities to learn about and experience teaching mathematical reasoning in K-8 teacher preparation programs – e.g., case studies, planning and teaching sample lessons to peer or K-8 students in field experiences – is desirable (Ball, 1990; Conklin et al., 2006; Ryve, 2011; Truxaw & DeFranco, 2008). At my institution, for example, the class experience shared here is supplemented by a case study about mathematical reasoning and discourse (see Rickard, 2014) and multiple lesson planning experiences in a mathematics methods course. The internship year also involves teaching experiences that include mathematical reasoning as one criteria on which prospective teachers are assessed and coached by mentor teachers. Ongoing inservice development of subject matter knowledge and pedagogical content knowledge of mathematics further supports teaching mathematical reasoning, as K-8 teachers need deep and flexible understanding of mathematics and ways of teaching it to be confident and effective in engaging their own students with mathematical reasoning (Rickard, 1996, 1998; Vacc & Bright, 1999; Wilson et al., 1987).

Finally, as an addendum to our experience with mathematical reasoning, one of my students asked at the beginning of the next class if we were finished with the theory that we had revised, or if we had more work to do. Recall that we had revised the theory/conjecture to be: *If the perimeter of a figure increases by increasing all its dimensions, then the area of the figure increases*. I asked the class what they thought and opinion was divided – some students thought the revised conjecture was fine and others thought we could probably refine it further. My (admittedly impromptu) response was that this is another facet of mathematical reasoning and how mathematics works – sometimes a problem or conjecture is set aside, other times some mathematicians continue to work on it. As a class, I said, we needed to continue on with new material, but anyone with an idea for a refinement of the conjecture could share it with the class for consideration. This response generated many nods of consent and we continued with the next topic.[10] I thought to myself about how Lakatos (1989) viewed the way mathematics is developed, stating that "Naïve conjectures and naïve concepts are superseded by improved conjectures (theorems) and concepts (proof-generated or theoretical concepts) growing out of the method of proofs and refutations" (p. 91). Mathematical reasoning is a process, not necessarily a finished product, and the same can be said for learning about teaching mathematics in general and teaching mathematical reasoning in particular.

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[10] Two students did suggest a revised conjecture a couple of days later: *If the perimeter of a figure increases so that it is similar to the original figure, then the area also increases*. The students said using similarity in the conjecture made sense because it linked to proportionality used in the earlier discussion. The class approved.

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